Hamiltonian Traffic Dynamics in Microfluidic-Loop Networks

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Recent microfluidic experiments revealed that large particles advected in a fluidic loop display long-range hydrodynamic interactions. However, the consequences of such couplings on the traffic dynamics in more complex networks remain poorly understood. In this Letter, we focus on the transport of a finite number of particles in one-dimensional loop networks. By combining numerical, theoretical, and experimental efforts, we evidence that this collective process offers a unique example of Hamiltonian dynamics for hydrodynamically interacting particles. In addition, we show that the asymptotic trajectories are necessarily reciprocal despite the microscopic traffic rules explicitly break the time-reversal symmetry. We exploit these two remarkable properties to account for the salient features of the effective three-particle interaction induced by the exploration of fluidic loops.

The long-range nature of the hydrodynamic interactions is responsible for fascinating collective phenomena in nonequilibrium suspensions, such as the velocity fluctuations of sedimenting particles [1], and the emergence of coherent structures in isotropic suspensions of active particles [2]. However, in confined geometries, the walls screen exponentially the correlations of the particle velocity [3]. Hence, no collective traffic phenomenon can occur when dilute suspensions flow in ducts having a width comparable to the particle size. Nonetheless, recent microfluidic experiments in channels including a loop, revealed a rich variety of collective dynamics, such as multiperiodic and multistable traffic patterns [4–10]. These experimental observations have been rationalized on the basis of two empirical rules [6]: (i) as a particle enters a loop, it takes the branch in which the flow rate is maximal, and (ii) the particles partly obstruct the branch in which they flow. Consequently, the particle velocity at a node is a function of the particle positions in the whole loop, thereby inducing localized but long-range hydrodynamic interactions. So far, most of the research on microfluidic traffic flows have been dedicated to the transport through a single fluidic loop fed at a constant rate by a continuous droplet or bubble stream.

In this letter, we investigate the dynamics of a finite number of particles cruising in an extended loop-network, see Fig. 1. We henceforth focus on the three-body problem. This setup is the basic building block to model the traffic dynamics of dilute suspensions (the case of two particles being trivial). Our primary idea is to consider the traffic through a single loop as a scattering process, which maps the distances \( \lambda(n) = [A_1(n), A_2(n)] \) between the three particles entering the loop \( n \) into a new set of distances \( \lambda(n+1) = S[A(n)] \), where \( S \) is the scattering map. The transport through the entire network is then conceived as a discrete dynamical system, for which the loop index \( n \) stands for the time variable. From this perspective, we first evidence that, remarkably, the asymptotic traffic dynamics is Hamiltonian. To the best of our knowledge, this is the only system of hydrodynamically interacting particles, for which an Hamiltonian description exists. Moreover, we show that the dynamics is asymptotically invariant upon time-reversal symmetry despite the microscopic traffic rules are explicitly nonreciprocal. We exploit these two features to account for the geometrical and the dynamical properties of the scattering-map \( S \). We close this paper, by comparing our theoretical predictions to microfluidic experiments. A quantitative agreement is found without any free fitting parameter.

We use a well established framework to model the traffic dynamics in a fluidic network made of a chain of \( N \) identical loops [7]. Precisely, it consists of four rules, which have proven to yield excellent agreement with the experiments [6–10]: (i) The flow state of the fluid in the network is given by the analogous of the Kirchhoff laws. (ii) The particles are supposed to have a constant mobility coefficient. Therefore, we identify the fluid and the particle velocities. (iii) When it reaches a vertex, a particle takes the branch where the fluid velocity is the higher. Note that this empirical rule, observed on deformable particles, is explicitly nonreciprocal. (iv) The particles partly obstruct the channels in which they journey. Precisely, the hydrodynamic resistances, expressed in unit length, are given by

\[
\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \ldots
\]

FIG. 1. Sketch of our theoretical and experimental setup: three droplets are advected in a one-dimensional microfluidic-loop network. \( \lambda_j (A_j) \) is the distance between the two rightmost (leftmost) droplets.
where \( n_{l1} \) are the numbers of particles advected in the upper and in the lower branches, respectively. The \( L_{l1} \) represent the branches’ length, and \( L_p \) is the constant additional resistance induced by a single droplet. It follows that the particle velocity in the upper branch is

\[
v_l = v \frac{L_{l1}(n_1)}{L_{l1}(n_1) + L_l(n_1)},
\]

FIG. 2 (color online). Numerical results obtained for \( a = 10/9 \). (a)- Basins of attraction of the \( S \) map. The gray dots correspond to initial distances yielding stationary asymptotic dynamics (zone C). The red dots converge to closed periodic orbits. Clogging parameter: \( c = 10/9 \). (b)- Dots: Superimposed asymptotic trajectories for \( c = 20/9 \). The red (gray) trajectories are enclosed in region B (C). The 8 polygons correspond to the 8 trafficking scenarios introduced on page 3. (c)- Dots: Superimposed asymptotic trajectories, same parameters as in (a). The red (gray) trajectories are enclosed in regions A and B (in region C). (d)- Close-up of the edge of one island, same parameters as in (c).
However, the global symmetry of the phase portrait with respect to $\lambda_1 = \lambda_2$ is preserved; see Fig. 2(c). Several closed orbits are destabilized into separatrix and island chains centered on stable $p$-periodic points. Trajectories with $p = 15$ are clearly seen in Fig. 2(c). We systematically observed a hierarchy of island chains, as exemplified in the close-up shown in Fig. 2(d). The inner part of the largest islands clearly include island chains as well. They are separated by large chaotic regions, which also exist at the largest scale of the phase portrait, though they are much less extended.

We close this numerical section with the first main result of this Letter. Remarkably, all the features of the phase portrait are the hallmarks of Hamiltonian mappings, despite the traffic dynamics is a driven dissipative process. We shall note that fluid mechanics offers other examples of Hamiltonian descriptions for advected particles. However, these models have so far been restricted to noninteracting passive tracers in bidimensional and incompressible fluids, for which the stream function readily provides an effective Hamiltonian [13]. The system, we consider here, does not make impossible the use of a stream function as an effective hydrodynamic coupling between the particles, through the loop. The system transit from one occupation state to an other, when a particle reaches one of the two vertices of the loop. To make this definition clearer, we write explicitly the sequences corresponding to the two scenarios, which chiefly rule the asymptotic dynamics. The scenario $A = \{(0, 1), (1, 1), (1, 2), (1, 1), (0, 1)\}$ is exemplified by the experimental pictures in Fig. 4(a). Three particles journey simultaneously in the loop, thereby inducing a change in the particle distances. Scenario $B = \{(0, 1), (1, 1), (1, 0), (1, 1), (0, 1)\}$, the loop is explored at most by two particles simultaneously. The other six traffic patterns are explicitly given, and sketched, in [12]. Practically, $S$ is a piecewise map, which has a different analytical expression, $S_X$, for each scenario. We first locate the regions of the phase plane in which each scenario prevails. To do so, using Eq. (1), we compute the 5 times, $t_X^{(i)}$, $i = 1 \ldots 5$, at which a particle reaches a vertex. The linearity of the Kirchhoff laws, implies that the $t_X^{(i)}$ are linear functions of $\lambda_1$ and $\lambda_2$. Consequently, the region corresponding to the scenario $X$ is a polygon defined by the inequalities: $t_X^{(i)}(\lambda_1, \lambda_2) < t_X^{(i+1)}(\lambda_1, \lambda_2)$. The 8 polygons tile the phase plane as illustrated in Fig. 2. We can then calculate the two distances $S_X(\lambda(n)) = (\lambda_1(n + 1), \lambda_2(n + 1))$ by computing the time intervals, which separate the exit of two subsequent particles from the loop, and multiplying it by the fluid velocity outside the loops, $v = v_1 + v_1$. Again, the Kirchhoff laws require the $S_X$ to be affine functions of the interparticle distances: $\lambda(n + 1) = M_X \cdot \lambda(n) + L_X$, where the $M_X$ and the $L_X$ are constant matrices and constant vectors. Their exact but lengthy expressions are given in the supplemental material [12].

We now exploit these analytical results to give a more physical insight on the geometrical and dynamical properties of the traffic dynamics. Firstly, by superimposing the numerical trajectories on the eight regions of the phase plane, we notice that the asymptotic orbits are enclosed only in the union of the polygons $A$ and $B$, Fig. 2. Moreover, the orbits that are enclosed in only one of those two regions are ellipses. To account for these observations, we compute the eigenvalues and the determinant of the $M_X$. Independently of the values of $a$ and $c$, $M_X$ is area preserving, $\det M_X = 1$, in these two regions. Beyond our numerical observations, this central result unambiguously proves that the three-particle dynamics is Hamiltonian in $A$ and $B$. Furthermore, a tedious calculation proved that the eigenvalues of $M_A$ and $M_B$ are two complex conjugate numbers; see [12]. Consequently, the orbits are necessarily self-similar ellipses centered on a unique fixed point, when solely enclosed in $A$ or $B$, in agreement with our numerical results, Fig. 2. In addition, the system necessarily converges toward the three Hamiltonian regions $A$, $B$, and $C$ (region $C$ corresponds to the trivial case $S_C = \emptyset$). Indeed, $|\det M_X|$ takes only two different expressions elsewhere. $|\det M_X| = a(1 + c)/(a + c)$, in regions $X = D, E, F, G$ and $|\det M_X| = (1 + c)(a + c)/(a + 2c)$, in region $X = H$. In both cases we verify that $|\det M_X| > 1$, as $a < 1 + c$. This implies that, asymptotically, the corresponding maps yield a continuous increase of $|\lambda_1|$ and $|\lambda_2|$. Therefore, as these maps are defined only in polygons having a finite width, we conclude that the system escapes from these regions as the particles flow through the loops.
We also infer from this observation, that the largest invariant curve is tangent to one of the boundary lines of the polygon $A \cup B$; see Fig. 2.

A second and important generic result is that the asymptotic traffic dynamics is time reversible. We now outline the demonstration of this result, which we use to account for the symmetry of the phase portrait with respect to the $\lambda_1 = \lambda_2$ direction. In this context, time reversal corresponds to the permutation of the interparticle distances: $\mathcal{T} : (\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1)$. Indeed, the last two particles that exit a loop correspond to the first two entering particles when reversing the flow. Saying that $S$ is time-reversible thus translates into $\mathcal{T} S \mathcal{T} S = \mathbb{1}$. This relation is obviously met along 1D trajectories enclosed in only one of the two regions $A$ or $B$. The corresponding traffic scenarios indeed correspond to palindromic sequences of occupation states. The same result can be also directly checked, by computing $(\mathcal{T} S_{X})^{2}$, where $X = A, B$, using the analytic expressions of the affine maps given in [12]. This identity is also satisfied for trajectories overlapping the polygons $A$ and $B$ as well. The reason for this is that $\mathcal{T} S_{X}(\lambda) \in X$ for the $\lambda$s belonging to the invariant curves of the region $X = A, B$. The demonstration of this last result is tedious. It is detailed in the supplemental document [12]. In order to show that the global symmetry of the phase portrait reflects the invariance upon time-reversal symmetry, let us consider a 1D orbit that crosses the symmetry line of $\mathcal{T}$, in the straight parts of the channel, we deduce the experimental value of $L_{d} = 1.2 \pm 0.25$ mm from Eq. (1). This makes possible a direct comparison between our experimental and our theoretical results, without any free fitting parameter. The evolution of $\lambda_{1}(n)$ and $\lambda_{2}(n)$ are plotted in Fig. 4(b). The gray value of each point codes for the traffic scenario we observed experimentally. Though, the fine structure of the phase portrait cannot be probed with a 20-loops network, an excellent agreement between our experimental and theoretical results is found, when considering the three generic features of the asymptotic-dynamics: (i) The two asymptotic-dynamics schemes. The distances oscillate around a fixed point when $\lambda_{1}$, $\lambda_{2} < \lambda_{\text{max}}$ and the traffic scenarios are of type $A$ or $B$ only. In contrast, when $\lambda_{1}$, $\lambda_{2} > \lambda_{\text{max}}$, we only observed small and nonpredictable variations of the $\lambda_{j}$. Complete freezing was never observed due to fluctuations in the droplet size, inducing differences in the droplets’ mobility. (ii) Our model perfectly predicts the location of the straight boundaries between the different traffic regions. (iii) The experimental phase portrait is symmetric with respect to the $\lambda_{1} = \lambda_{2}$ direction.

In conclusion, combining experimental, numerical and theoretical tools, we have provided a comprehensive description of the three-body traffic dynamics. We expect that the generalization of our approach to coupled elementary traffic maps should provide a useful toolbox to design functional microfluidic devices.

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[12] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.108.034501 for the description of all the traffic scenarios, and the form of the associated maps. We also provide the exact expression of the eigenvalues of the affine maps $A$ and $B$. Finally, we present a rigorous demonstration of the identity $TSTS = I$.